Connection Aspects of the Non-linear Schrödinger Equation

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Some geometrical features of the non-linear Schrödinger equation are studied and it is described how the Schrödinger equation can be obtained from the non-linear scattering equations. The $SL(2, \mathbb{R})$-valued elements of the matrix of the non-linear scattering problem are interpreted as matrix-valued forms. We calculate the curvature form with respect to a basis of the Lie algebra. If the curvature form equals zero then we obtain the non-linear Schrödinger equation.

Wahlquist and Estabrook have given a geometric approach [1] and they find a prolongation structure for non-linear partial differential equations in two independent variables. They have applied their method to the non-linear Schrödinger equation and they discussed the relationship between the pseudopotentials and the inverse scattering method (ISM) of Zakharov and Shabat [2], and Ablowitz, Kaup, Newell and Segur [3]. Sasaki [4] has shown that the ISM is given by a completely integrable pfaffian system and that the equations of Ablowitz, Kaup, Newell and Segur can be described by a pseudospherical surface. Furthermore, Sasaki [5] has given pseudopotentials for equations of Ablowitz, Kaup, Newell und Segur.

We give a further example of the Lie algebraic method to study the geometrical aspects of the non-linear Schrödinger equation. First we cite the scattering equations (ISM) and some formulae [6]. We use the (ISM) in the form

$$\frac{\partial}{\partial x}\begin{pmatrix}\varphi^1 \\ \varphi^2\end{pmatrix} = \begin{pmatrix} \eta & q(x,t) \\ r(x,t) & -\eta \end{pmatrix}\begin{pmatrix}\varphi^1 \\ \varphi^2\end{pmatrix}, \quad (1)$$

where the time evolution of the functions $\varphi^1(x,t)$ and $\varphi^2(x,t)$ are given by

$$\frac{\partial}{\partial t}\begin{pmatrix}\varphi^1 \\ \varphi^2\end{pmatrix} = \begin{pmatrix} A(x,t;\eta) & B(x,t;\eta) \\ C(s,t;\eta) & -A(x,t;\eta) \end{pmatrix}\begin{pmatrix}\varphi^1 \\ \varphi^2\end{pmatrix}. \quad (2)$$

The coefficients A, B and C are functions of x and time t. The quantity η is the eigenvalue of the

scattering problem and we assume $\eta \neq \eta(t)$. The system (1) and (2) can be rewritten in **matrix notation**. We have

$$\frac{\partial \varphi^k}{\partial x^j} + \sum_{p=1}^{2} \Omega_{pj}^k \cdot \varphi^p = 0, \quad (3)$$

where $j, k = 1, 2$ and $x^1 = x$, $x^2 = t$. The $\varphi^k(x,t)$ are the components of a two-component field on the principal bundle $P(M, G)$. The Ω_{pj}^k are given by the components of the matrix in (1) and (2). The theory of Cartan-Ehresmann connections (see [7]) generalizes the Gaussian curvature of a Riemannian 2-dimensional manifold. The curvature form is given by the exterior covariant derivative of the 1-form ω on $P(M, G)$ with the values in a finite-dimensional vector space V in the form

$$\Theta = \nabla \omega = d\omega \circ h, \quad (4)$$

where Θ is a g-valued 2-form. The $(p+1)$-form $\nabla \omega$ is given by

$$\nabla \omega(X_1, \ldots, X_{p+1}) = d\omega(h X_1, \ldots, h X_{p+1}), \quad (5)$$

and $h: T_p(P(M,G)) \to S_p$ the projection of T_p (tangential space) onto its horizontal subspace S_p. The space of vertical vectors $V_p = T_p \Theta S_p$ lies tangential to the fibre.

In a previous paper [6] we have expressed the curvature form Θ on bundles $P(M, G)$ in the form

$$\Theta = \sum_{i=1}^{3} d\omega^i \otimes X_i + \frac{1}{2} \sum_{i,j=1}^{3} (\omega^i \wedge \omega^j) \otimes [X_i, X_j], \quad (6)$$

where $\omega^k (k = 1, 2, 3)$ are arbitrary one-forms and $[X_p, X_q]$ is the commutator of the quantities X_k. The two-form Θ is called the curvature of the connection. The $\{X_k\}_{k=1}^{3}$ are a basis of the Lie algebra $g = SL(2, \mathbb{R})$, and we choose

$$X_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$
$$X_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \quad (7)$$

In view of (3) we can write for the 1-forms

$$\omega^1 = -(\eta\,dx + A\,dt),$$
$$\omega^2 = -(q\,dx + B\,dt),$$
$$\omega^3 = -(r\,dx + C\,dt). \quad (8)$$

If we rewrite the curvature form (6) with (7) and (8) then we obtain the expression

$$\Theta = (qC - rB - A_x) \cdot dx \wedge dt \otimes X_1$$
$$+ (2\eta B - 2qA + q_t - B_x)$$
$$\cdot dx \wedge dt \otimes X_2$$
$$+ (-2\eta C + 2rA + r_t - C_x)$$
$$\cdot dx \wedge dt \otimes X_3 . \qquad (9)$$

If we choose

$$r = -q^* = -u^*, \quad A = 2i\eta^2 + i|u|^2,$$
$$B = iu_x + 2i\eta u, \quad C = iu_x^* - 2i\eta u^*, \quad (10)$$

then we obtain

$$\Theta = (-2ui|u|^2 + u_t - iu_{xx})\, dx \wedge dt \otimes X_2$$
$$+ (-2u^*i|u|^2 - u_t^* - iu_{xx}^*)$$
$$\cdot dx \wedge dt \otimes X_3 . \qquad (11)$$

If $\Theta = 0$ we obtain

$$iu_t + u_{xx} + 2|u|^2 u = 0,$$
$$-iu_t^* + u_{xx}^* + 2|u|^2 u^* = 0, \qquad (12)$$

for the real and imaginary part of the non-linear Schrödinger equation. Moreover, from the condition $\Theta = 0$ we conclude that

i) ω satisfies the Maurer-Cartan structural equation

$$d\omega + \tfrac{1}{2}[\omega, \omega] = 0,$$

ii) the connection in $P(M, G)$ is flat.

To summarize: We have used the Lie algebraic method to give a geometrical interpretation of the non-linear Schrödinger equation. We see that ω satisfies the structure equation of Maurer-Cartan, which implies that the canonical flat connection has zero curvature. Therefore we can say that the non-linear Schrödinger equation can be deduced from the fact that the $SL(2, \mathbb{R})$ connection associated with the scattering equation has zero curvature. The Bäcklund transformation for the non-linear Schrödinger equation is given by Lamb [8] in a different way by using the method of pseudo-potentials [1].

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